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Jackson's Theorem for Compact Connected Lie Groups

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We prove that for $f \in E = C(G)$ or $L^p(G)$, $1 \leqslant p < \infty$, where G is any compact connected Lie group, and for $n \geqslant 1$, there is a trigonometric polynomial t_n on G of degree $\leqslant n$ so that $\|f - t_n\|_E \leqslant C_r \omega_r(n^{-1}, f)$. Here $\omega_r(t, f)$ denotes the rth modulus of continuity of f. Using this and sharp estimates of the Lebesgue constants recently obtained by Giulini and Travaglini, we obtain "best possible" criteria for the norm convergence of the Fourier series of f. © 1988 Academic Press, Inc.

1. INTRODUCTION

Let E be a Banach space of periodic functions on $\mathbf{R}$, let $f \in E$, and let $n \geqslant 1$ be an integer. A basic problem in approximation theory is to estimate the quantity

$$\mathscr{E}_n(f) = \inf_t \{\|f - t\|_E\},$$

the infimum being taken over all trigonometric polynomials t of degree at most n. Jackson's Theorem is the fundamental "direct theorem" here; it asserts that if the rth derivative $f^{(r)}$ exists in E (in the appropriate sense) and if E is suitable, then $\mathscr{E}_n(f) \leqslant C_r n^{-r} \omega_1(n^{-1}, f^{(r)}) = o(n^{-r})$ (see [8]). More precise versions of Jackson's Theorem provide estimates $\mathscr{E}_n(f) \leqslant C_r \omega_r(n^{-1}, f)$ for any $f \in E$, where $\omega_r(t, f)$ is the rth modulus of continuity of f.

Jackson's Theorem extends in a straightforward way to periodic functions of k variables (i.e., functions on the group $\mathbf{T}^k$) and it is natural to ask whether it also applies to functions on nonabelian groups. In this paper, we prove that Jackson's Theorem is true for any compact connected Lie group.

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THEOREM. *Let $G \neq \{1\}$ be any compact connected Lie group. Let E denote one of the spaces $C(G)$ or $L^p(G)$, $1 \leq p < \infty$, and let $r \geq 1$ be an integer. Then there is a constant C_r and for each integer $n \geq 1$ there is a central trigonometric polynomial K_n of degree $\leq n$ such that*

$$\|f - K_n * f\|_E \leq C_r \omega_r\left(\frac{1}{n}, f\right)$$

for each $f \in E$.

Johnen [6] proved this theorem in the special case $r = 2$, but our method is quite different from his. The kernels K_n are related to the $\tilde{\Phi}_n$ of [3], but even more to those used in [8, 9] in proving the $\mathbf{T}^k$ case.

As an application of our theorem, we use the sharp estimates for the Lebesgue constants recently obtained by Giulini and Travaglini [4] to give "best possible" criteria for the norm convergence of Fourier series of functions on G.

2. NOTATION

We shall basically adopt the notation of [1], but see also [5] for some terminology.

Let G be a compact connected Lie group, let T be a fixed maximal torus in G and let $\mathbf{g}$ and $\mathbf{t}$ ($\subset \mathbf{g}$) denote their respective Lie algebras. We choose an inner product $\langle\ ,\ \rangle$ on $\mathbf{g}$ which is invariant under the adjoint action of G on $\mathbf{g}$. This provides inner products on $\mathbf{t}$ and its dual $\mathbf{t}^*$ which are invariant under the action of the Weyl group $W = N(T)/T$. Let $I = \{H \in \mathbf{t} : \exp H = 1\}$ be the integral lattice and let $I^* = \{\gamma \in \mathbf{t}^* : \gamma(H) \in \mathbf{Z}$ for all $H \in I\}$ be the lattice of integral forms. For $r \geq 0$, a *trigonometric polynomial on T of degree $\leq r$* is a linear combination of the characters $\theta_\gamma : \exp H \mapsto e^{2\pi i \gamma(H)}$ of T, where $\gamma \in I^*$ and $\|\gamma\| \leq r$. Here $\|\ \|$ is the norm on $\mathbf{t}^*$ induced by $\langle\ ,\ \rangle$. Note that the degree of a trigonometric polynomial need not be an integer.

Let $R \subset I^*$ be the set of real roots of G. The characters θ_α, $\alpha \in R$, are the nontrivial weights of the adjoint representation Ad of G on the complexification $\mathbf{g}_C$ of $\mathbf{g}$. Let $\{\alpha_1, ..., \alpha_l\}$ be a basis for R and let $K \subset \mathbf{t}^*$ be the corresponding Weyl chamber and R_+ the corresponding set of positive roots. The dual object $\hat{G}$ of G may be identified with $\bar{K} \cap I^*$ (see [1, p. 242]). If $\gamma \in \bar{K} \cap I^*$, let $\mathcal{T}_\gamma(G)$ denote the space of trigonometric polynomials corresponding to γ [5, p. 5]. A trigonometric polynomial which is central (i.e., a class function) is a linear combination of the characters χ_γ associated to $\gamma \in \bar{K} \cap I^*$. For $r \geq 0$, a *trigonometric polynomial on G of degree $\leq r$* is a linear combination of functions in $\mathcal{T}_\gamma(G)$ for $\|\gamma\| \leq r$.

The above definitions of degree depend of course on the choice of $\langle\ ,\ \rangle$. When G is semisimple, we may choose $\langle\ ,\ \rangle$ to be $-\psi$, where ψ is the Killing form [1, p. 214]. When G is also simply connected, functionals $\lambda_1, ..., \lambda_l \in \mathbf{t}^*$ satisfying $2\langle \lambda_j, \alpha_k \rangle / \langle \alpha_k, \alpha_k \rangle = \delta_{jk}$ for $j, k = 1, ..., l$, are integral forms, and $\bar{K} \cap I^* = \{n_1 \lambda_1 + \cdots + n_l \lambda_l : 0 \leqslant n_1, ..., n_l \in \mathbf{Z}\}$ ([1, p. 255]).

Let $f \in E$, where $E = C(G)$ or $L^p(G)$, $1 \leqslant p < \infty$. For any integer $r \geqslant 1$ and for $t > 0$, let

$$\omega_r(t, f) = \sup\{\|\Delta^r_{\exp X} f\|_E : X \in \mathbf{g} \text{ and } \|X\| \leqslant t\}$$

be the rth modulus of continuity of f. Here

$$(\Delta^r_h f)(x) = \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} f(h^{-j}x) \tag{1}$$

for $x, h \in G$. We note that $\omega_r(t, f) = \sup\{\|\Delta^r_h f\|_E : h \in G \text{ and } d(1, h) \leqslant t\}$, where d is the geodesic distance on G induced by $\langle\ ,\ \rangle$ on $\mathbf{g}$ (see [7, Eq. (1.1.7)]). We need here only the inequality $\omega_r(\lambda t, f) \leqslant (1+\lambda)^r \omega_r(t, f)$, valid for $\lambda > 0$ ([7, Eq. (1.1.10)]).

3. Proof of the Theorem

We first need two straightforward lemmas.

Lemma 1. *The restriction map $\phi \mapsto \phi_{|T}$ is a degree-preserving isomorphism of the space of central trigonometric polynomials on G onto the space of trigonometric polynomials on T invariant under the action of the Weyl group.*

Proof. In view of [1, Corollary IV(2.7) and Proposition VI(2.1)], we need only check that $\deg \phi_{|T} = \deg \phi$. Let $0 \neq \phi = \sum_{\gamma \in \bar{K} \cap I^*} a_\gamma \chi_\gamma$ be a central trigonometric polynomial on G of degree r. Then $r = \max\{\|\gamma\| : a_\gamma \neq 0\}$. By [1, Proposition VI(2.6)], $\phi_{|T}$ can be expressed in terms of symmetric sums $S(\gamma)$,

$$\phi(\exp H) = \sum_\gamma a_\gamma \left[S(\gamma)(H) + \sum_j n_{\gamma,j} S(\lambda_{\gamma,j})(H) \right], \tag{2}$$

where $\lambda_{\gamma,j} \in \bar{K} \cap I^*$ and $\lambda_{\gamma,j} < \gamma$ for each γ, j. By [1, Proposition VI(2.4ii)], $\|\lambda_{\gamma,j}\| \leqslant \|\gamma\| \leqslant r$ for each j if $a_\gamma \neq 0$. If we pick $\gamma_0 \in \bar{K} \cap I^*$ such that $a_{\gamma_0} \neq 0$ and $\|\gamma_0\| = r$ and γ_0 is maximal with respect to the partial order $\leqslant$, then the term $a_{\gamma_0} S(\gamma_0)(H)$ will appear in (2) (i.e., will not be cancelled by any term $S(\lambda_{\gamma',j})$). Thus $\deg \phi_{|T} = r$.

LEMMA 2. *Let $s \geqslant 1$ be an integer. Then, writing* $A(H)$ for $A(\varrho)(H)$, *where* $\varrho = \frac{1}{2}\sum_{\alpha \in R_+} \alpha$ [1, Chap. VI, Sect. 1], *we have*

$$|A(sH)/A(H)|^2 = \psi_s(\exp H), \qquad \text{for} \quad H \in t,$$

for a central trigonometric polynomial ψ_s *on G of degree* $2(s-1)\|\varrho\|$.

Proof. By [1, Eq. VI(1.5)], we have

$$A(H) = e^{2\pi i \varrho(H)} \prod_{\alpha \in R_+} (1 - e^{-2\pi i \alpha(H)})$$

and so

$$\begin{aligned} |A(sH)/A(H)|^2 &= \prod_{\alpha \in R} \frac{1 - \theta_\alpha(\exp sH)}{1 - \theta_\alpha(\exp H)} \\ &= \prod_{\alpha \in R} \left(\sum_{k=0}^{s-1} \theta_\alpha(t)^k \right), \qquad \text{for} \quad t = \exp H \\ &= \psi_s(t), \text{ say.} \end{aligned}$$

Now ψ_s is a trigonometric polynomial on T which is invariant under the action of the Weyl group, since $A(H)$ is alternating. Clearly ψ_s is a linear combination of θ_γ's with γ of the form $\sum_{\alpha \in R_+} n_\alpha \alpha$ and $|n_\alpha| \leqslant s-1$. So $|A(sH)/A(H)|^2$ is a linear combination of symmetric sums $S(\gamma)$ with $\gamma \in \bar{K} \cap I^*$ and $\gamma \leqslant 2(s-1)\varrho$. Thus ψ_s has degree $2(s-1)\|\varrho\|$ by [1, Prop. VI(2.4ii)].

Proof of the Theorem.

For some $k \geqslant 1$ there is a diffeomorphism $h: \exp H \mapsto (e^{2\pi i \gamma_1(H)}, ..., e^{2\pi i \gamma_k(H)})$ of T onto $\mathbf{T}^k$. This induces a linear isomorphism $h_*: \mathbf{t} \to \mathbf{R}^k$, namely $H \mapsto (\gamma_1(H), ..., \gamma_k(H))$. Let $Q = h_*^{-1}((-\frac{1}{2}, \frac{1}{2}]^k)$, so that $\exp_{|Q}$: $Q \to T$ is a bijection. Since $\mathbf{t} \cong \mathbf{R}^k$, we can define the Schwartz space $\mathscr{S}(\mathbf{t})$ of smooth functions on $\mathbf{t}$ all of whose derivatives tend to zero rapidly at infinity (see, e.g., [10, Sect. I.3]). Let m denote Lebesgue measure on $\mathbf{t}$, normalized so that $\int_t e^{-\pi\|H\|^2}\, dH = 1$, where we write dH instead of $dm(H)$.

Let ψ be a smooth function on $\mathbf{t}$ such that $\psi(H) = 0$ if $\|H\| \geqslant c$, where $c = \min\{\|w\varrho - \varrho\| : w \in W, w \neq 1\}$ $(= \min\{\|\alpha\| : \alpha \in R\})$, and such that $\psi(0) = m(Q)$. Replacing ψ by $\psi^\#$, where $\psi^\#(H) = |W|^{-1} \sum_{w \in W} \psi(wH)$, we may suppose that ψ is W-invariant. Define $\phi: \mathbf{t} \to \mathbf{C}$ by $\phi(H) = \int_t \psi(H')\, e^{2\pi i \langle H, H' \rangle}\, dH'$. Then ϕ has the following properties:

(i) $\phi \in \mathscr{S}(\mathbf{t})$;

(ii) ϕ is W-invariant;

(iii) $\hat{\phi}(H) = 0$ if $\|H\| \geqslant c$; and

(iv) $\int_t |A(H)|^2\, \phi(H)\, dH = |W|\, m(Q)$, where $\hat{\phi}(H)=\int_t \phi(H')\, e^{-2\pi i\langle H,H'\rangle}\, dH'$ is the Fourier transform of ϕ. The first three properties clearly hold, since ϕ (because of the normalization of m) is the inverse Fourier transform of ψ, and W acts orthogonally on t. Also $A(H)=\sum_{w\in W}\det(w)\, e^{2\pi i\varrho(wH)}$ and so

$$|A(H)|^2=|W|+\sum_{w,w'\in W,w\neq w'}\det(ww')\, e^{2\pi i(w\varrho-w'\varrho)(H)}$$

$$=\sum_{\gamma\in F} a_\gamma e^{2\pi i\gamma(H)},$$

where $F\subset I^*$ ([1, p. 207]) is finite, with $a_0=|W|$ and $\|\gamma\|=\|w\varrho-w'\varrho\|\geqslant c$ for some w, $w'\in W$ with $w\neq w'$ if $0\neq\gamma\in F$. Thus $\int_t |A(H)|^2\phi(H)\,dH=\sum_{\gamma\in F} a_\gamma\psi(-\gamma)=a_0\psi(0)=|W|\,m(Q)$ (identifying t and t*).

We now use the Poisson summation formula to construct for each integer $s\geqslant 1$ a central trigonometric polynomial $\tilde{\phi}_s$ on G of degree $<cs$. Let

$$\phi_s(H)=s^k\sum_{H'\in I}\phi(s(H+H')).$$

Since $\phi\in\mathscr{S}(\mathbf{t})$, this defines a smooth W-invariant function ϕ_s on t such that $\phi_s(H+H')=\phi_s(H)$ for $H'\in I$. This corresponds to a W-invariant function $\tilde{\phi}_s$ on T. For $\gamma\in I^*$, we have

$$(\tilde{\phi}_s)\hat{}\,(\theta_\gamma)=\int_T \tilde{\phi}_s(t)\,\overline{\theta_\gamma(t)}\,dt$$

$$=m(Q)^{-1}\int_Q \phi_s(H)\, e^{-2\pi i\gamma(H)}\,dH$$

$$=s^k m(Q)^{-1}\int_t \phi(sH)\, e^{-2\pi i\gamma(H)}\,dH$$

$$=m(Q)^{-1}\hat{\phi}(\gamma/s)$$

$$=0,\qquad \text{if}\quad \|\gamma\|\geqslant cs.$$

Thus $\tilde{\phi}_s$ is a W-invariant trigonometric polynomial of degree $<cs$. Now use Lemma 1.

For an integer $n\geqslant 2(2\|\varrho\|+c)\,r!$, let m be the integer part of $n/2(2\|\varrho\|+c)r!$, and let $n'=r!(m+1)$. Define

$$K_n=\sum_{j=1}^{r}(-1)^{j-1}\binom{r}{j}\psi_{n'/j}\tilde{\phi}_{n'/j}.$$

By Lemma 2 and the above steps, K_n is a central trigonometric polynomial on G of degree at most

$$2(n'-1)\,\|\varrho\|+n'c<(2\,\|\varrho\|+c)(m+1)r!\leqslant n.$$

By the Weyl integral formula (with η as in [1, Lemma VI(1.8)]),

$$\begin{aligned}
&(K_n * f)(y)\\
&\quad = \int_G K_n(x)\, f(x^{-1}y)\, dx\\
&\quad = |W|^{-1} \int_T \eta(t)\, K_n(t) \left\{ \int_G f(gt^{-1}g^{-1}y)\, dg \right\} dt\\
&\quad = C \int_Q |A(H)|^2\, K_n(\exp H)\, J(H)\, dH\\
&\quad = C \sum_{j=1}^{r} (-1)^{j-1} \binom{r}{j} \int_Q |A(n'H/j)|^2\, \phi_{n'/j}(H)\, J(H)\, dH\\
&\quad = C \sum_{j=1}^{r} (-1)^{j-1} \binom{r}{j} \int_t |A(n'H/j)|^2 (n'/j)^k\, \phi(n'H/j)\, J(H)\, dH,
\end{aligned}$$

where $C = (m(Q)\, |W|)^{-1}$ and $J(H) = \int_G f(g \exp(-H)\, g^{-1}y)\, dg$. Changing variables, this equals

$$\begin{aligned}
&C \int_t |A(H)|^2\, \phi(H)\\
&\qquad \times \left\{ \int_G \sum_{j=1}^{r} (-1)^{j-1} \binom{r}{j} f(g \exp(-jH/n')\, g^{-1}y)\, dg \right\} dH\\
&\quad = f(y) + (-1)^{r-1} C \int_t |A(H)|^2\, \phi(H)\\
&\qquad \times \left\{ \int_G (\Delta^r_{\exp(\mathrm{Ad}(g)H/n')} f)(y)\, dg \right\} dH.
\end{aligned}$$

Thus, by Minkowski's inequality,

$$\begin{aligned}
&\|f - K_n * f\|_E\\
&\quad \leqslant C \int_t |A(H)|^2\, |\phi(H)| \left\{ \int_G \|\Delta^r_{\exp(\mathrm{Ad}(g)H/n')} f\|_E\, dg \right\} dH\\
&\quad \leqslant C \int_t |A(H)|^2\, |\phi(H)|\, \omega_r(\|H\|/n', f)\, dH\\
&\quad \leqslant C_r \omega_r(1/n, f),
\end{aligned}$$

as $\|\mathrm{Ad}(g)H/n'\| = \|H\|/n' \leqslant 2(2\,\|\varrho\| + c)\, \|H\|/n$, where

$$C_r = C \int_t |A(H)|^2\, |\phi(H)|\, (1 + 2(2\,\|\varrho\| + c)\, \|H\|)^r\, dH.$$

Remark. If we modify the definition of $\omega_r(t, f)$ by writing xh^j instead of $h^{-j}x$ in (1), the theorem is still valid because $K_n * f = f * K_n$.

4. Convergence of Fourier Series

In this section G denotes a semisimple compact connected Lie group and $E = C(G)$ or $L^1(G)$. For $f \in E$ and $n \geqslant 1$, $s_n f = \sum_{\gamma \in C_n} d_\gamma \chi_\gamma * f$ is called the nth *spherical* [resp. *polyhedral*] partial sum of the Fourier series $\sum_{\gamma \in \bar{K} \cap I^*} d_\gamma \chi_\gamma * f$ of f if $C_n = \{\gamma \in \bar{K} \cap I^* : \|\gamma + \varrho\| \leqslant n\}$ [resp. $C_n = \{\gamma \in \bar{K} \cap I^* : \gamma \leqslant n\omega\}$, where $\omega \in K \cap I^*$ is fixed]. Giulini and Travaglini [4] have recently obtained sharp estimates of the so-called *Lebesgue constants* $\sup\{\|s_n f\|_E : \|f\|_E \leqslant 1\} = \|\sum_{\gamma \in C_n} d_\gamma \chi_\gamma\|_1$ in both these cases. In the case of spherical partial sums, they showed that

$$c_1 n^{(d-1)/2} \leqslant \left\| \sum_{\gamma \in C_n} d_\gamma \chi_\gamma \right\|_1 \leqslant c_2 n^{(d-1)/2}$$

holds for $d = \dim G$ and for suitable constants $c_1, c_2 > 0$, while for polyhedral sums similar inequalities hold, but with $(d-1)/2$ replaced by $|R_+|$. We can now state a refinement of the Proposition in [4].

Proposition. *Let G be a semisimple compact connected Lie group and let $E = C(G)$ or $L^1(G)$.*

(a) *If $f \in E$ and $\omega_r(t, f) = o(t^{(d-1)/2})$ as $t \to 0$ for some integer $r \geqslant (d-1)/2$, then the spherical partial sums $s_n f$ converge to f in E.*

(b) *There exists $F \in E$ such that $\omega_r(t, F) = O(t^{(d-1)/2})$ as $t \to 0$ but for which $s_n F$ does not converge to F in E. In fact, if $0 \leqslant s < (d-1)/2$ is an integer, we may choose $F \in E^{(s)}$ with $\omega_{r-s}(t, F^{(s)}) = O(t^{(d-1)/2 - s})$ for all $r \geqslant (d-1)/2$.*

The corresponding result holds for polyhedral partial sums with $(d-1)/2$ replaced by $|R_+|$ throughout.

Proof. The proof is a simple modification of the proof of Theorem B in [2], and is therefore omitted.

Remark. The restriction $r \geqslant (d-1)/2$ on r in the Proposition is necessary because $\omega_r(t, f) = o(t^r)$ as $t \to 0$ implies that f is constant. In fact, if $M = \sup\{d(1, h) : h \in G\}$ and if $0 < t \leqslant M$, then (for $E = C(G)$),

$$|f(g) - f(1)| \leqslant \omega_1(M, f) \leqslant (1 + M/t)\,\omega_1(t, f) \leqslant 2Mt^{-1}\omega_1(t, f).$$

This proves the assertion when $r = 1$. For general r, apply inequality (3.4) in [6].

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